

Question 9

On $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, consider

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Let S be the outward-oriented unit sphere.

Tasks

1. Compute the flux through S directly.
2. Explain why zero divergence away from the origin does not permit applying the Divergence Theorem to the full unit ball.
3. Reconcile the result using a punctured ball.

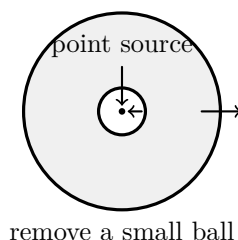
Question 9 – Solution

Strategy. The nonzero flux is caused by a singularity inside the sphere; remove a small ball so the theorem's hypotheses hold.

Step 1: Direct flux On the unit sphere, $\mathbf{F} = \mathbf{n}$, so

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S 1 \, dS = 4\pi.}$$

Although $\nabla \cdot \mathbf{F} = 0$ for $r > 0$, the field is undefined at the origin and is not smooth throughout the full ball.



Step 2: Puncture the ball Remove the ball $r < \varepsilon$. On the shell $\varepsilon \leq r \leq 1$, the divergence is zero, so the total outward flux is zero.

Step 3: Track both boundaries The outer flux is 4π . On $r = \varepsilon$, the shell's outward normal is $-\mathbf{e}_r$, while $\mathbf{F} = \mathbf{e}_r/\varepsilon^2$. Thus

$$\Phi_{r=\varepsilon} = -\frac{1}{\varepsilon^2}(4\pi\varepsilon^2) = -4\pi.$$

Therefore $4\pi - 4\pi = 0$, consistent with the theorem on the punctured region.

Verification The inner flux remains -4π as $\varepsilon \rightarrow 0^+$, recording the enclosed point source rather than disappearing.