

Question 7

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Its closed boundary is oriented outward. For

$$\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle,$$

compute the net outward flux and verify the contributions from the two boundary pieces.

Tasks

1. Describe the cylindrical-coordinate bounds for E .
2. Apply the Divergence Theorem.
3. Compute the plane and paraboloid fluxes separately.

Question 7 – Solution

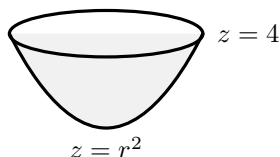
Strategy. The divergence equals 1, so the net flux is the volume; the direct check tests the downward orientation on the paraboloid.

Step 1: Bounds and volume The surfaces meet at $r = 2$, and

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad r^2 \leq z \leq 4.$$

Since $\nabla \cdot \mathbf{F} = 1$,

$$\Phi_{\text{net}} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta = \boxed{8\pi}.$$



Step 2: Top plane On $z = 4$, the outward normal is $\mathbf{k}$, so

$$\Phi_{\text{top}} = \iint_{r \leq 2} 4 \, dA = 16\pi.$$

Step 3: Paraboloid The solid lies above the paraboloid, so its outward vector element is downward:

$$\mathbf{n} \, dS = \langle 2x, 2y, -1 \rangle \, dA.$$

Thus

$$\Phi_{\text{paraboloid}} = \iint_{r \leq 2} -r^2 \, dA = -2\pi \int_0^2 r^3 \, dr = -8\pi.$$

The direct total is $16\pi - 8\pi = 8\pi$.

Verification The sign of the lower contribution is negative because the field points upward while the lower outward normal points downward.