

Question 6

Let E be the first-octant tetrahedron $x + y + z \leq 1$, and let $S = \partial E$ be outward oriented. For

$$\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle,$$

compute the total outward flux.

Tasks

1. Find the divergence.
2. Evaluate the needed coordinate moments over the tetrahedron.
3. Check the answer directly on the slanted face.

Question 6 – Solution

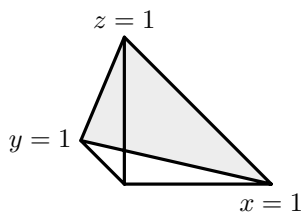
Strategy. Symmetry makes the three first moments equal; the coordinate-plane faces also provide a short direct check.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z.$$

The tetrahedron has volume $1/6$ and centroid $(1/4, 1/4, 1/4)$, so

$$\iiint_E x \, dV = \iiint_E y \, dV = \iiint_E z \, dV = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}.$$

**Step 2: Apply the theorem**

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{24} \right) = \frac{1}{4}.}$$

Step 3: Direct check On each coordinate face the corresponding field component is zero, so those fluxes vanish. On $z = 1 - x - y$, the outward vector element is $\langle 1, 1, 1 \rangle dA$, giving

$$\iint_D (x^2 + y^2 + (1 - x - y)^2) \, dA = \frac{1}{4}.$$

Verification The direct slanted-face integral agrees with the divergence result and is positive, as expected in the first octant.