

Question 10

On $\mathbb{R}^3$ minus the z -axis, define

$$\mathbf{F}(x, y, z) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right\rangle.$$

Let C be the unit circle in $z = 0$, counterclockwise from above.

Tasks

1. Compute $\nabla \times \mathbf{F}$ where the field is defined.
2. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.
3. Explain why applying Stokes to the full disk is invalid, and reconcile the result using a punctured disk.

Question 10 – Solution

Strategy. The apparent contradiction “zero curl but nonzero circulation” signals a failed hypothesis: the field is singular on every full disk spanning C .

Step 1: Curl and direct integral Differentiation gives

$$\nabla \times \mathbf{F} = \mathbf{0} \quad (x^2 + y^2 > 0).$$

On C , let $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = 2\pi.$$



Step 2: Locate the failed hypothesis The field is undefined at the origin, which lies on the full spanning disk. Thus $\mathbf{F}$ does not have continuous derivatives on an open set containing that disk, and Stokes' Theorem cannot be applied there.

Step 3: Use an annulus Remove a radius- ε disk. On the resulting annulus the curl is zero, so Stokes gives

$$0 = \oint_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_{\text{inner, clockwise}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi - 2\pi.$$

Verification The inner boundary carries exactly the circulation needed to account for the puncture; no contradiction remains.