

Question 9

Let S consist of the top and four side faces of the unit cube $0 \leq x, y, z \leq 1$, but not the bottom face. Give S the outward orientation. Its boundary C is the rim of the missing bottom square. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem on the composite surface.

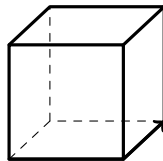
Tasks

1. Determine the induced direction of C as viewed from above.
2. Compute the curl flux through the five faces.
3. Verify the result using the missing bottom square.

Question 9 – Solution

Strategy. The curl is vertical, so the four vertical faces contribute zero; only the top face remains.

Step 1: Boundary direction The outward orientation on the four sides induces counterclockwise traversal of the bottom rim when viewed from above.



C : rim of the missing bottom

Step 2: Flux through S Since

$$\nabla \times \mathbf{F} = \mathbf{k},$$

its dot product with every horizontal side-face normal is zero. On the top, the outward normal is $\mathbf{k}$, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0 + \iint_{[0,1]^2} 1 \, dA = 1.$$

Step 3: Replace the surface The upward-oriented bottom square has the same counterclockwise boundary C . Applying Stokes to that single square also gives $\iint 1 \, dA = 1$.

Verification Using the outward normal $-\mathbf{k}$ on the missing bottom face would induce the opposite boundary direction; it is not the compatible replacement for the stated C .