

Question 7

Let C be the unit circle in $z = 0$, counterclockwise from above. Two surfaces span it: the upper hemisphere S_1 and the paraboloid $S_2 : z = x^2 + y^2 - 1, x^2 + y^2 \leq 1$, both oriented to induce the given direction. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

compare the curl fluxes through S_1 and S_2 .

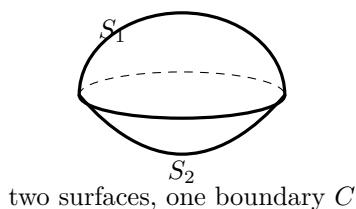
Tasks

1. Determine compatible orientations on both surfaces.
2. Evaluate one common boundary circulation or curl flux.
3. Explain why the two fluxes must agree.

Question 7 – Solution

Strategy. Use Stokes' Theorem as a surface-independence statement: each compatible curl flux equals the same circulation around C .

Step 1: Orient the surfaces Use the outward orientation on the upper hemisphere and the upward orientation on the paraboloid. Both induce counterclockwise motion on C viewed from above.



Step 2: Compute the common value Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

we may use the upward unit disk D spanning C :

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2\pi.$$

Consequently,

$$\boxed{\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS = 2\pi.}$$

Step 3: Explain Stokes' Theorem ties curl flux to the oriented boundary, not to the particular spanning surface, provided the field is smooth throughout the relevant surfaces.

Verification Both surfaces project once onto the unit disk with positive vertical orientation, so integrating the vertical curl 2 directly would also give 2π on each.