

Question 6

Use Stokes' Theorem to derive an area formula for a positively oriented simple closed curve C in the xy -plane. Then apply it to the ellipse

$$C : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0.$$

Tasks

1. Choose a vector field with constant vertical curl.
2. Derive $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
3. Evaluate the formula for the ellipse.

Question 6 – Solution

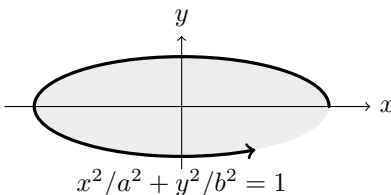
Strategy. Encode planar area as the curl flux of a field whose vertical curl is 1.

Step 1: Choose the field Let

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle.$$

Then $\nabla \times \mathbf{F} = \mathbf{k}$. If D is the region enclosed by positively oriented C , Stokes gives

$$\text{area}(D) = \iint_D 1 \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$



Step 2: Parametrize the ellipse Use

$$x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi.$$

Then

$$x \, dy - y \, dx = ab(\cos^2 t + \sin^2 t) \, dt = ab \, dt.$$

Therefore

$$\boxed{\text{area}(D) = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab}.$$

Verification Setting $a = b = R$ recovers the circle area πR^2 , including the correct scale and orientation sign.