

Question 2

Let C be the triangular boundary of the first-octant portion of the plane $x + y + z = 1$. Orient C consistently with the normal having positive components. For

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl and an oriented vector area element.
2. Evaluate the surface integral over the projected triangle.
3. Verify the result by integrating along the three edges.

Question 2 – Solution

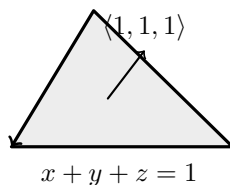
Strategy. Use the triangular plane itself; its graph vector element is constant and points in the prescribed direction.

Step 1: Curl and surface element

$$\nabla \times \mathbf{F} = \langle 1, 1, 1 \rangle.$$

Writing $z = 1 - x - y$ over $D = \{x, y \geq 0 : x + y \leq 1\}$ gives

$$\boxed{\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy}.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle dA \\ &= 3 \text{ area}(D) = \boxed{\frac{3}{2}}. \end{aligned}$$

Step 3: Edge check The orientation follows

$$(1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1) \rightarrow (1, 0, 0).$$

Parametrizing each edge by $0 \leq t \leq 1$ gives integrands $1 - t$, $1 - t$, and $1 - t$, respectively. Hence the three line integrals are each $1/2$, totaling $3/2$.

Verification Two successive directed edge vectors have cross product parallel to $\langle 1, 1, 1 \rangle$, confirming the boundary orientation.