

Question 9

A plane patch over the unit square has two parametrizations

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, q + 2p \rangle.$$

For $\mathbf{F} = \langle 1, 0, 0 \rangle$, compare the flux values produced by the ordered cross products of the two parametrizations.

Tasks

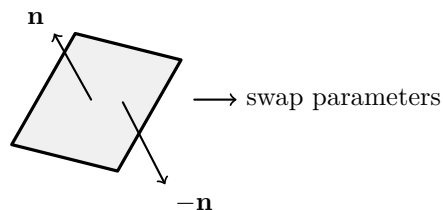
1. Show that $\mathbf{s}$ swaps the parameters of $\mathbf{r}$.
2. Compute both signed flux integrals.
3. Explain how to make the two computations represent the same geometric orientation.

Question 9 – Solution

Strategy. Swapping parameters reverses the cross product, so a vector surface integral detects a sign change that a scalar surface integral would not.

Step 1: Compare cross products Because $\mathbf{s}(p, q) = \mathbf{r}(q, p)$, the images are the same. However,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$



Step 2: Compute signed fluxes Since $\mathbf{F} = \langle 1, 0, 0 \rangle$ and both parameter domains have area 1,

$$\iint \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv = \boxed{-1},$$

whereas

$$\iint \mathbf{F} \cdot (\mathbf{s}_p \times \mathbf{s}_q) dp dq = \boxed{1}.$$

Step 3: Match orientation If the orientation from $\mathbf{r}_u \times \mathbf{r}_v$ is prescribed geometrically, then the $\mathbf{s}$ calculation must use $-(\mathbf{s}_p \times \mathbf{s}_q)$. It then also gives -1 .

Verification The two raw answers are negatives, exactly as required because the ordered normals are negatives. Flux is invariant under reparametrization only after the geometric orientation is held fixed.