

Question 8

The Möbius band can be parametrized by

$$\mathbf{r}(u, v) = \left\langle \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \cos u, \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \sin u, \frac{v}{2} \sin \frac{u}{2} \right\rangle,$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Tasks

1. Verify the seam identification $\mathbf{r}(0, v) = \mathbf{r}(2\pi, -v)$.
2. Compare the transverse tangent direction at the two identified edges.
3. Explain why a globally oriented flux integral is not defined on the band.

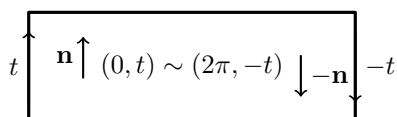
Question 8 – Solution

Strategy. Follow a normal direction once around the band and test whether it returns consistently at the identified seam.

Step 1: Identify the seam Direct substitution gives

$$\mathbf{r}(0, v) = \left\langle 1 + \frac{v}{2}, 0, 0 \right\rangle = \mathbf{r}(2\pi, -v).$$

Thus the left edge at height v is glued to the right edge at height $-v$.



the seam reverses the transverse direction

Step 2: Compare transverse tangents Along the centerline $v = 0$,

$$\mathbf{r}_v(0, 0) = \left\langle \frac{1}{2}, 0, 0 \right\rangle, \quad \mathbf{r}_v(2\pi, 0) = \left\langle -\frac{1}{2}, 0, 0 \right\rangle.$$

Also $\mathbf{r}_u(0, 0) = \mathbf{r}_u(2\pi, 0) = \langle 0, 1, 0 \rangle$. Hence

$$(\mathbf{r}_u \times \mathbf{r}_v)(0, 0) = \left\langle 0, 0, -\frac{1}{2} \right\rangle, \quad (\mathbf{r}_u \times \mathbf{r}_v)(2\pi, 0) = \left\langle 0, 0, \frac{1}{2} \right\rangle.$$

Step 3: Conclude The two parameter points represent the same surface point, yet continuation around the strip reverses the normal. Therefore no continuous global unit normal exists, and

a global oriented flux integral over the Möbius band is not defined.

Verification Cutting the band at the seam permits a local choice of normal, but gluing the seam back forces opposite choices, exposing the obstruction.