

Question 7

Let S be the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. Compute the flux of the constant field $\mathbf{F} = \langle 0, 0, 1 \rangle$ across S .

Tasks

1. Express the hemisphere as a graph and find its upward vector element.
2. Evaluate the flux using projection onto the xy -plane.
3. State the result for the opposite orientation.

Question 7 – Solution

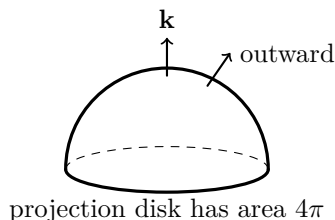
Strategy. A constant vertical field dotted with the upward graph element leaves exactly the projected area element.

Step 1: Graph orientation Write

$$z = g(x, y) = \sqrt{4 - x^2 - y^2}$$

over the disk $D : x^2 + y^2 \leq 4$. The outward orientation on the upper hemisphere is upward, so

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA.$$



Step 2: Project Since $\mathbf{F} = \mathbf{k}$,

$$\mathbf{F} \cdot \mathbf{n} dS = \mathbf{k} \cdot \langle -g_x, -g_y, 1 \rangle dA = dA.$$

Thus

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D dA = \pi(2^2) = 4\pi}.$$

Step 3: Reverse orientation With the inward orientation the vector element changes sign, so the flux is $\boxed{-4\pi}$.

Verification The graph derivatives become unbounded at the equator, but their components are annihilated by the dot product with $\mathbf{k}$. Truncating to radius $R < 2$ and taking $R \rightarrow 2^-$ yields the same result.