

Question 2

Let S be the paraboloid patch $z = x^2 + y^2$ above $x^2 + y^2 \leq 1$, oriented upward. For

$$\mathbf{F}(x, y, z) = \langle x, y, 1 \rangle,$$

compute the flux across S .

Tasks

1. Derive the upward vector surface element.
2. Evaluate the flux in polar coordinates.
3. Explain how positive and negative contributions cancel.

Question 2 – Solution

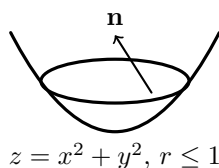
Strategy. Use the upward graph element $\langle -g_x, -g_y, 1 \rangle dA$ and then exploit radial symmetry.

Step 1: Vector area element For $g(x, y) = x^2 + y^2$,

$$\mathbf{n} dS = \langle -2x, -2y, 1 \rangle dA.$$

Thus

$$\mathbf{F} \cdot \mathbf{n} dS = (-2x^2 - 2y^2 + 1) dA = (1 - 2r^2) dA.$$



Step 2: Integrate

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^1 (1 - 2r^2) r dr d\theta \\ &= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^1 = \boxed{0}. \end{aligned}$$

Step 3: Locate the cancellation The normal component is positive for $r < 1/\sqrt{2}$ and negative for $r > 1/\sqrt{2}$. The inner contribution is

$$2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^{1/\sqrt{2}} = \frac{\pi}{4},$$

while the outer annulus contributes $-\pi/4$.

Verification The exact cancellation explains why zero flux does not mean the field is tangent everywhere.