

Question 9

Let S be the entire unbounded paraboloid $z = x^2 + y^2$. Determine whether the improper scalar surface integral

$$\iint_S \frac{e^{-z}}{\sqrt{1+4z}} dS$$

converges, and evaluate it if it does.

Tasks

1. Truncate the surface above disks $x^2 + y^2 \leq R^2$.
2. Reduce the truncated surface integral to one variable.
3. Take the improper limit and justify convergence.

Question 9 – Solution

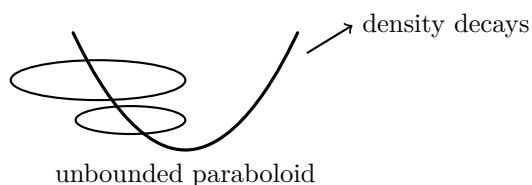
Strategy. On this paraboloid, the denominator in the density is exactly the graph-area factor, leaving a Gaussian radial integral.

Step 1: Truncate Let S_R be the part above $0 \leq r \leq R$. For $z = r^2$,

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$

Therefore the density times the area element simplifies to

$$\frac{e^{-r^2}}{\sqrt{1 + 4r^2}} \sqrt{1 + 4r^2} r dr d\theta = e^{-r^2} r dr d\theta.$$



Step 2: Evaluate the truncation

$$\begin{aligned} \iint_{S_R} \frac{e^{-z}}{\sqrt{1 + 4z}} dS &= 2\pi \int_0^R e^{-r^2} r dr \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^R \\ &= \pi(1 - e^{-R^2}). \end{aligned}$$

Step 3: Take the limit The integrand is nonnegative and the truncated values increase to a finite limit:

$$\boxed{\iint_S \frac{e^{-z}}{\sqrt{1 + 4z}} dS = \lim_{R \rightarrow \infty} \pi(1 - e^{-R^2}) = \pi.}$$

Verification The omitted tail equals πe^{-R^2} , which tends to zero and quantifies the convergence.