

Question 8

Find the area of the helicoid patch

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Compute the tangent vectors and surface element.
2. Evaluate the resulting nonpolynomial integral.
3. Check regularity and the scale of the area.

Question 8 – Solution

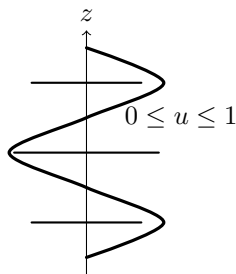
Strategy. Compute the cross product in the given parameters; its magnitude depends only on u , so the v -integration separates.

Step 1: Surface element

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle, \quad \boxed{dS = \sqrt{1 + u^2} \, du \, dv}.$$

**Step 2: Integrate**

$$\begin{aligned} \text{area}(S) &= 2\pi \int_0^1 \sqrt{1 + u^2} \, du \\ &= 2\pi \left[\frac{1}{2} \left(u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2}) \right) \right]_0^1 \\ &= \boxed{\pi \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}. \end{aligned}$$

Step 3: Check The cross product never vanishes because its first two components have squared sum 1. Also $1 \leq \sqrt{1 + u^2} \leq \sqrt{2}$, so

$$2\pi \leq \text{area}(S) \leq 2\pi\sqrt{2},$$

and the exact result lies between these bounds.