

Question 5

On the sphere $S : x^2 + y^2 + z^2 = 4$, evaluate the scalar surface integral

$$\iint_S z^2 dS.$$

Then find the average value of z^2 over the sphere.

Tasks

1. Choose spherical parameters and compute dS .
2. Evaluate the integral exactly.
3. Use symmetry to check the average value.

Question 5 – Solution

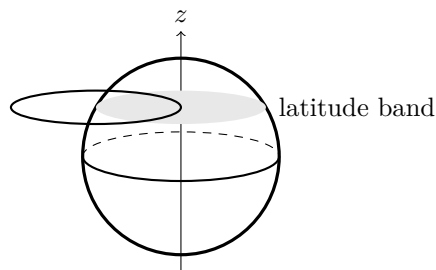
Strategy. In spherical parameters, z and dS separate into a polar factor and an azimuthal factor.

Step 1: Parametrize For radius 2, use

$$\mathbf{r}(\phi, \theta) = \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Then

$$z = 2 \cos \phi, \quad dS = 4 \sin \phi \, d\phi \, d\theta.$$



Step 2: Integrate

$$\begin{aligned} \iint_S z^2 \, dS &= \int_0^{2\pi} \int_0^\pi 16 \cos^2 \phi \sin \phi \, d\phi \, d\theta \\ &= 32\pi \int_0^\pi \cos^2 \phi \sin \phi \, d\phi \\ &= 32\pi \left[\frac{-\cos^3 \phi}{3} \right]_0^\pi = \boxed{\frac{64\pi}{3}}. \end{aligned}$$

Because $\text{area}(S) = 4\pi(2^2) = 16\pi$, the average is

$$\boxed{\text{Avg}_S(z^2) = \frac{4}{3}}.$$

Verification Rotational symmetry gives equal averages of x^2 , y^2 , z^2 . Their sum is always 4, so each average must be $4/3$.