

Question 4

Let S be the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 3.$$

Tasks

1. Parametrize S and derive dS .
2. Compute the surface area.
3. Evaluate $\iint_S \frac{1}{z} dS$ and its average value on S .

Question 4 – Solution

Strategy. Use height as the radius parameter. The factor of u in dS cancels against the weight $1/z$ in the second integral.

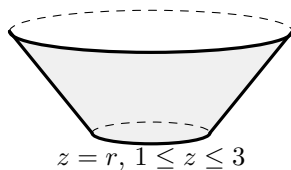
Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

The tangent vectors have cross-product magnitude

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{2},$$

so $\boxed{dS = u\sqrt{2} \, du \, dv}.$



Step 2: Area

$$\text{area}(S) = \int_0^{2\pi} \int_1^3 u\sqrt{2} \, du \, dv = \boxed{8\pi\sqrt{2}}.$$

Step 3: Weighted integral

Since $z = u$,

$$\iint_S \frac{1}{z} \, dS = \int_0^{2\pi} \int_1^3 \frac{1}{u} u\sqrt{2} \, du \, dv = \boxed{4\pi\sqrt{2}}.$$

Its average value is therefore

$$\boxed{\frac{4\pi\sqrt{2}}{8\pi\sqrt{2}} = \frac{1}{2}}.$$

Verification On the frustum, $1/3 \leq 1/z \leq 1$; the average $1/2$ lies in this interval.