

Question 3

A thin surface occupies the lateral cylinder

$$S: x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

with surface density $\delta(x, y, z) = 1 + z$.

Tasks

1. Parametrize S and compute its surface element.
2. Find the total mass.
3. Find the center of mass, using symmetry where appropriate.

Question 3 – Solution

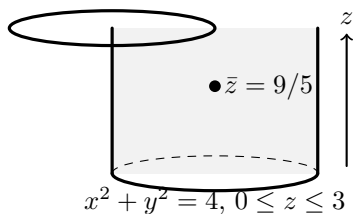
Strategy. Parametrize by angle and height. Rotational symmetry immediately locates two coordinates of the center of mass.

Step 1: Geometry

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

Since $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = 2$,

$$dS = 2 \, d\theta \, dz.$$

**Step 2: Mass**

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^3 (1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[z + \frac{z^2}{2} \right]_0^3 = \boxed{30\pi}. \end{aligned}$$

Step 3: Center of mass Symmetry gives $\bar{x} = \bar{y} = 0$. The first moment about the xy -plane is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^3 z(1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[\frac{z^2}{2} + \frac{z^3}{3} \right]_0^3 = 54\pi. \end{aligned}$$

Thus

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{9}{5} \right).$$

Verification Because density increases with height, $\bar{z} = 1.8$ is correctly above the geometric midpoint 1.5 and remains inside $0 \leq z \leq 3$.