

Question 2

Let S be the part of the paraboloid $z = x^2 + y^2$ above the unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Express dS in polar coordinates.
2. Set up and evaluate $\iint_S z \, dS$.
3. Check the radial substitution and the scale of the result.

Question 2 – Solution

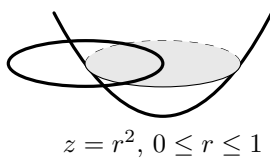
Strategy. Use the graph formula first, then exploit radial symmetry so the surface integral becomes one one-variable integral.

Step 1: Surface element With $g(x, y) = x^2 + y^2$,

$$dS = \sqrt{1 + 4x^2 + 4y^2} dA.$$

In polar coordinates, $z = r^2$ and $dA = r dr d\theta$, so

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$



Step 2: Integrate

$$\iint_S z dS = 2\pi \int_0^1 r^3 \sqrt{1 + 4r^2} dr.$$

Let $t = 1 + 4r^2$. Then $r dr = dt/8$ and $r^2 = (t - 1)/4$, giving

$$\begin{aligned} \iint_S z dS &= \frac{\pi}{16} \int_1^5 (t^{3/2} - t^{1/2}) dt \\ &= \frac{\pi}{16} \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_1^5 \\ &= \boxed{\frac{\pi(25\sqrt{5} + 1)}{60}}. \end{aligned}$$

Verification The integrand is nonnegative. Also $0 \leq z \leq 1$, so the answer must not exceed the surface area; numerically it is about 2.98, consistent with that bound.