

Question 10

A torus has major radius 3 and tube radius 1, centered about the z -axis.

Tasks

1. Construct a parametrization using two angles.
2. Compute an outward-pointing normal vector.
3. Prove the parametrization is regular everywhere and describe its seam identifications.

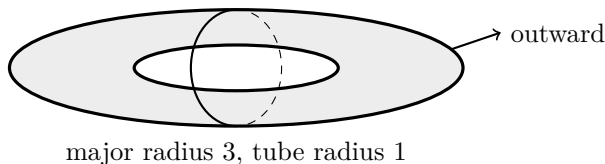
Question 10 – Solution

Strategy. First parametrize a radius-1 circle centered 3 units from the axis, then rotate it about the z -axis.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle (3 + \cos v) \cos u, (3 + \cos v) \sin u, \sin v \rangle,$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.



Step 2: Compute a normal Let $A = 3 + \cos v$. Then

$$\mathbf{r}_u = \langle -A \sin u, A \cos u, 0 \rangle,$$

$$\mathbf{r}_v = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle.$$

Their cross product is

$$\mathbf{r}_u \times \mathbf{r}_v = A \langle \cos u \cos v, \sin u \cos v, \sin v \rangle,$$

which points from the tube's center circle toward the surface, hence outward.

Step 3: Regularity and seams The unit vector in brackets has magnitude 1, while $A = 3 + \cos v \geq 2$. Thus the cross product never vanishes. The edges $u = 0, 2\pi$ are identified, as are $v = 0, 2\pi$.

Verification The distance from the z -axis is $3 + \cos v$ and the height is $\sin v$, so $(\sqrt{x^2 + y^2} - 3)^2 + z^2 = 1$, the intended torus.