

Question 8

Two parametrizations are

$$\mathbf{r}(u, v) = \langle u, v, u + v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, p + q \rangle.$$

Tasks

1. Show that they have the same image.
2. Compute the ordered normal from each parametrization.
3. Determine whether they induce the same or opposite orientation.

Question 8 – Solution

Strategy. Recognize that the second parametrization swaps the two parameter roles, then compare ordered cross products.

Step 1: Compare images Both satisfy

$$x + y = z.$$

Given (u, v) for $\mathbf{r}$, choosing $(p, q) = (v, u)$ gives $\mathbf{s}(v, u) = \mathbf{r}(u, v)$. Hence their images are the same plane.

Step 2: First normal

$$\mathbf{r}_u = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 1 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \boxed{\langle -1, -1, 1 \rangle}.$$

Step 3: Second normal

$$\mathbf{s}_p = \langle 0, 1, 1 \rangle, \quad \mathbf{s}_q = \langle 1, 0, 1 \rangle,$$

and therefore

$$\mathbf{s}_p \times \mathbf{s}_q = \boxed{\langle 1, 1, -1 \rangle}.$$

The normals are negatives, so the parametrizations induce opposite orientations.

Verification Swapping two parameters reverses the order in a cross product, and $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.