

Question 4

Consider the standard spherical parametrization

$$\mathbf{r}(\phi, \theta) = \langle 3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Tasks

1. Show that the image is the sphere of radius 3.
2. Compute an outward normal vector from the tangent vectors.
3. Identify where the parametrization fails to be regular or one-to-one.

Question 4 – Solution

Strategy. Verify the implicit equation, then use the ordered product $\mathbf{r}_\phi \times \mathbf{r}_\theta$.

Step 1: Identify the image

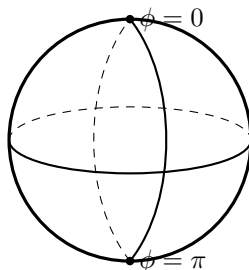
$$x^2 + y^2 + z^2 = 9 \sin^2 \phi + 9 \cos^2 \phi = 9.$$

Thus the image is $x^2 + y^2 + z^2 = 9$.

Step 2: Normal Differentiation and crossing give

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 9 \sin \phi \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle.$$

For $0 < \phi < \pi$, this is a positive multiple of the outward radial direction.



longitude collapses at the poles

Step 3: Degeneracies Its magnitude is $9 \sin \phi$, which vanishes at $\phi = 0$ and $\phi = \pi$. Every value of θ maps to the same pole there. Also $\theta = 0$ and $\theta = 2\pi$ describe the same meridian seam.

Verification Away from the poles, $9 \sin \phi > 0$, so the tangent vectors are independent and the patch is regular locally.