

Question 9

On $D = \mathbb{R}^3 \setminus \{\mathbf{0}\}$, define

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$ on D .
2. Compute or otherwise justify $\nabla \times \mathbf{F}$ on D .
3. Explain why the excluded origin must be stated.

Question 9 – Solution

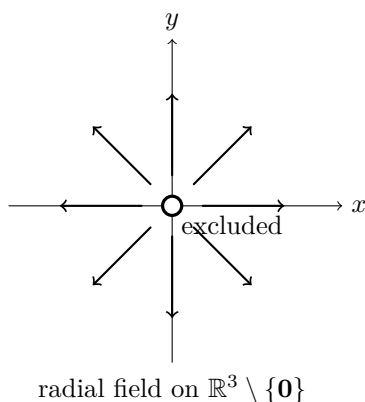
Strategy. Write $r = (x^2 + y^2 + z^2)^{1/2}$ and differentiate xr^{-3} , yr^{-3} , and zr^{-3} .

Step 1: Divergence Since

$$\frac{\partial}{\partial x}(xr^{-3}) = r^{-3} - 3x^2r^{-5},$$

with analogous formulas for y and z ,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= 3r^{-3} - 3(x^2 + y^2 + z^2)r^{-5} \\ &= 3r^{-3} - 3r^2r^{-5} = \boxed{0}, \quad r > 0.\end{aligned}$$



Step 2: Curl Because

$$\mathbf{F} = \nabla \left(-\frac{1}{r} \right)$$

on D , the curl-of-a-gradient identity gives

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0}}.$$

Step 3: Domain caveat The formula is undefined at the origin, and the derivatives used above do not exist there. The conclusions are pointwise statements on D , not statements at $\mathbf{0}$.

Verification Direct cross differentiation gives, for example, $R_y - Q_z = -3yzzr^{-5} - (-3yzzr^{-5}) = 0$, and the other two components cancel similarly.