

**Question 6**

For

$$f(x, y, z) = x^2y + yz^2 + xz,$$

verify directly that

$$\nabla \times (\nabla f) = \mathbf{0}.$$

**Tasks**

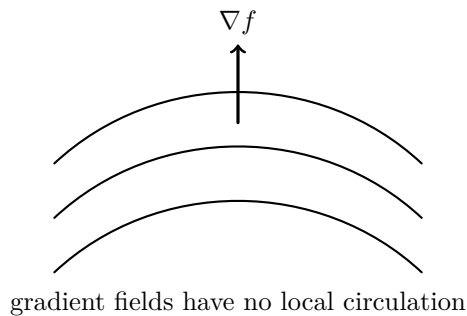
1. Compute  $\nabla f$ .
2. Compute its curl and show each cancellation.
3. State the smoothness principle behind the identity.

## Question 6 – Solution

**Strategy.** Differentiate once to obtain the gradient and again in the curl's cross-partial pairs.

### Step 1: Gradient

$$\nabla f = \langle 2xy + z, x^2 + z^2, 2yz + x \rangle.$$



**Step 2: Curl of the gradient** Let the gradient components be  $P, Q, R$ . Then

$$R_y - Q_z = 2z - 2z = 0,$$

$$P_z - R_x = 1 - 1 = 0,$$

$$Q_x - P_y = 2x - 2x = 0.$$

Therefore

$$\boxed{\nabla \times (\nabla f) = \mathbf{0}}.$$

**Step 3: General principle** Each cancellation is equality of a mixed-partial pair:

$$f_{zy} = f_{yz}, \quad f_{xz} = f_{zx}, \quad f_{yx} = f_{xy}.$$

Continuous second partial derivatives guarantee these equalities.

**Verification** The polynomial  $f$  has continuous derivatives of every order on  $\mathbb{R}^3$ , so there are no domain or differentiability exceptions to the identity here.