

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle.$$

Verify directly that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Tasks

1. Compute $\nabla \times \mathbf{F}$.
2. Take its divergence.
3. Explain why this is an instance of a general vector identity.

Question 5 – Solution

Strategy. Compute the curl componentwise, then differentiate each component with respect to its matching coordinate.

Step 1: Curl Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - 2xz, 0 - 2xy \rangle \\ &= \boxed{\langle -2yz, -2xz, -2xy \rangle}.\end{aligned}$$

Step 2: Divergence of the curl

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(-2yz) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(-2xy) \\ &= 0 + 0 + 0 = \boxed{0}.\end{aligned}$$

Step 3: General identity For a field with continuous second partial derivatives, expanding $\nabla \cdot (\nabla \times \mathbf{F})$ produces pairs of equal mixed partials with opposite signs. They cancel by equality of mixed partial derivatives.

Verification Each component of this particular curl omits the coordinate with respect to which divergence differentiates it, so all three terms vanish independently.