

Question 4

For constants a, b, d , consider

$$\mathbf{F}(x, y, z) = \langle ax - by, bx - ay, dz \rangle.$$

Classify all parameter triples (a, b, d) for which the field is both divergence-free and curl-free on $\mathbb{R}^3$.

Tasks

1. Compute the divergence.
2. Compute the curl.
3. Solve the simultaneous parameter conditions and verify the resulting family.

Question 4 – Solution

Strategy. Treat the scalar divergence and all three curl components as simultaneous identities.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = a + (-a) + d = d.$$

Thus divergence-free requires $d = 0$.

Step 2: Curl With $P = ax - by$, $Q = bx - ay$, and $R = dz$,

$$\nabla \times \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, b - (-b) \rangle = \langle 0, 0, 2b \rangle.$$

Thus curl-free requires $b = 0$.

Step 3: Classify The parameter a is unrestricted, so

$$\boxed{(a, b, d) = (a, 0, 0), \quad a \in \mathbb{R}}.$$

The resulting field is

$$\mathbf{F} = \langle ax, -ay, 0 \rangle.$$

Verification Its divergence is $a - a + 0 = 0$, and all cross derivatives vanish. Hence every field in the boxed family satisfies both conditions, including the zero field when $a = 0$.