

Question 9

Let C be a positively oriented, simple, piecewise smooth closed curve enclosing a region D . Derive the three area formulas

$$A = \oint_C x \, dy, \quad A = - \oint_C y \, dx, \quad A = \frac{1}{2} \oint_C (x \, dy - y \, dx).$$

Tasks

1. Choose P, Q for each of the first two formulas.
2. Derive the symmetric formula from Green's Theorem or by averaging.
3. Explain the orientation requirement.

Question 9 – Solution

Strategy. Make the Green's Theorem integrand $Q_x - P_y$ equal to 1 in three different ways.

Step 1: Formula using $x \, dy$ Choose $P = 0$, $Q = x$. Then $Q_x - P_y = 1$, so

$$\text{Area}(D) = \iint_D 1 \, dA = \boxed{\oint_C x \, dy}.$$

Step 2: Formula using $-y \, dx$ Choose $P = -y$, $Q = 0$. Again $Q_x - P_y = 1$, giving

$$\text{Area}(D) = \boxed{-\oint_C y \, dx}.$$

Step 3: Symmetric formula Choose $P = -y/2$ and $Q = x/2$. Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1,$$

and therefore

$$\boxed{\text{Area}(D) = \frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$

This is also the average of the first two formulas.

Verification Green's Theorem uses positive, counterclockwise orientation. Clockwise traversal negates all three line integrals, so a minus sign would be needed to recover positive geometric area.