

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay^2, x^2 \rangle.$$

The counterclockwise circulation around the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is -2 . Determine a .

Tasks

1. Express the circulation as a double integral.
2. Solve for a .
3. Verify uniqueness and the stated circulation.

Question 8 – Solution

Strategy. Green's Theorem turns the circulation condition into a linear equation for a .

Step 1: Form the integrand With $P = ay^2$ and $Q = x^2$,

$$Q_x - P_y = 2x - 2ay.$$

Thus

$$\begin{aligned}\oint_C \mathbf{F}_a \cdot d\mathbf{r} &= \int_0^1 \int_0^1 (2x - 2ay) dy dx \\ &= \int_0^1 (2x - a) dx = 1 - a.\end{aligned}$$

Step 2: Impose the condition

$$1 - a = -2 \quad \implies \quad \boxed{a = 3}.$$

Step 3: Verify For $a = 3$,

$$\int_0^1 \int_0^1 (2x - 6y) dy dx = 1 - 3 = -2,$$

exactly the specified counterclockwise circulation.

Uniqueness check The circulation $1 - a$ has nonzero coefficient -1 on a , so the equation has exactly one solution.