

Question 6

Use a Green's Theorem area formula to find the area enclosed by the cardioid

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq 2\pi,$$

traversed counterclockwise.

Tasks

1. Begin with $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
2. Show that $x \, dy - y \, dx = r^2 \, d\theta$.
3. Evaluate the resulting trigonometric integral.

Question 6 – Solution

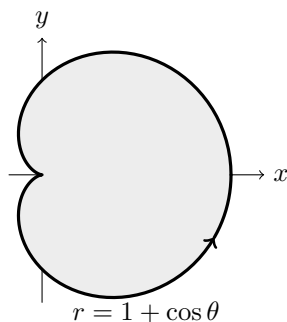
Strategy. Combine Green's area formula with polar boundary coordinates.

Step 1: Convert the differential With $x = r \cos \theta$ and $y = r \sin \theta$,

$$dx = (r' \cos \theta - r \sin \theta) d\theta, \quad dy = (r' \sin \theta + r \cos \theta) d\theta.$$

The rr' terms cancel, leaving

$$x \, dy - y \, dx = r^2 (\cos^2 \theta + \sin^2 \theta) d\theta = r^2 d\theta.$$



Step 2: Evaluate the area

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} (2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}. \end{aligned}$$

Verification The standard polar area formula is $\frac{1}{2} \int r^2 d\theta$, exactly what Green's Theorem produced. The nonnegative integrand confirms the positive area.