

Question 5

Let C be the counterclockwise boundary of the region between

$$y = x^2 \quad \text{and} \quad y = x, \quad 0 \leq x \leq 1.$$

Evaluate

$$\oint_C -y \, dx + x^2 \, dy.$$

Tasks

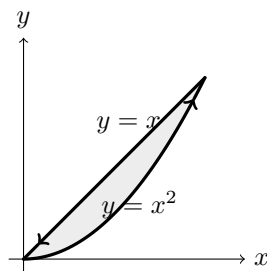
1. Describe the positively oriented boundary.
2. Set up the Green's Theorem double integral with correct bounds.
3. Evaluate and verify the sign.

Question 5 – Solution

Strategy. Replace the two curved boundary integrals with one Type I double integral over the enclosed region.

Step 1: Derivative difference Here

$$P = -y, \quad Q = x^2, \quad Q_x - P_y = 2x + 1.$$



Positive orientation travels from $(0, 0)$ to $(1, 1)$ along the lower parabola and returns along the upper line.

Step 2: Set up and evaluate

$$\begin{aligned} \oint_C P dx + Q dy &= \int_0^1 \int_{x^2}^x (2x + 1) dy dx \\ &= \int_0^1 (2x + 1)(x - x^2) dx \\ &= \int_0^1 (x + x^2 - 2x^3) dx \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \boxed{\frac{1}{3}}. \end{aligned}$$

Verification On $0 \leq x \leq 1$, both $2x + 1$ and the vertical thickness $x - x^2$ are nonnegative, so the positive result is required.