

Question 4

Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

The boundary consists of the outer circle and the inner circle.

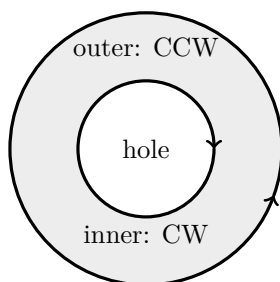
Tasks

1. Specify the positive orientation of both boundary components.
2. Apply Green's Theorem on the annulus.
3. Compute each circle's contribution and verify the cancellation.

Question 4 – Solution

Strategy. On a region with a hole, positive orientation is counterclockwise outside and clockwise around the hole.

Step 1: Orientation Traverse the radius-2 circle counterclockwise and the radius-1 circle clockwise. In both cases the annular region remains on the left.



Step 2: Apply Green's Theorem On D , the field is continuously differentiable and

$$Q_x - P_y = 0.$$

Therefore

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D 0 \, dA = \boxed{0}.$$

Step 3: Check each component On any counterclockwise circle centered at the origin, the integral is 2π , independent of radius. Hence

$$\int_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi, \quad \int_{C_{\text{inner}}} \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

because the inner circle is clockwise. Their sum is zero.

Verification If both circles were traversed counterclockwise, the total would be 4π ; that is not the positively oriented boundary of the annulus.