

Question 10

Classify the linear vector fields

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle,$$

where a, b, c, d are real constants.

Tasks

1. Find the necessary and sufficient condition for $\mathbf{F}$ to be conservative on $\mathbb{R}^2$.
2. Construct the general potential when the condition holds.
3. Under that condition, compute the work from $(0, 0)$ to (p, q) .

Question 10 – Solution

Strategy. Compare the constant cross partials, then integrate symbolically without assigning numerical coefficients.

Step 1: Classify Let $P = ax + by$ and $Q = cx + dy$. Then

$$P_y = b, \quad Q_x = c.$$

Because $\mathbb{R}^2$ is simply connected, equality is both necessary and sufficient. Hence

$$\boxed{\mathbf{F} \text{ is conservative exactly when } b = c.}$$

Step 2: Construct the potential Assume $b = c$. Integrating $f_x = ax + by$ gives

$$f = \frac{a}{2}x^2 + bxy + g(y).$$

Since $f_y = bx + g'(y) = bx + dy$, we have $g'(y) = dy$, so

$$\boxed{f(x, y) = \frac{a}{2}x^2 + bxy + \frac{d}{2}y^2 + C.}$$

Step 3: Compute the work The Fundamental Theorem for Line Integrals gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(p, q) - f(0, 0) = \boxed{\frac{a}{2}p^2 + bpq + \frac{d}{2}q^2}.$$

Verification Differentiating the potential gives $\langle ax + by, bx + dy \rangle$, which equals the original field precisely because $c = b$. The additive constant cancels from the work.