

Question 9

A field on $\mathbb{R}^2$ has the form

$$\mathbf{F}(x, y) = \langle 2xy + y, Q(x, y) \rangle.$$

It is conservative and satisfies $Q(0, y) = e^y$.

Tasks

1. Determine the most general Q allowed by the mixed-partial condition.
2. Use the boundary data to determine Q uniquely.
3. Find and verify a potential function.

Question 9 – Solution

Strategy. Integrate the compatibility equation for the unknown component, then use the supplied trace data to fix its arbitrary function.

Step 1: Compatibility equation With $P = 2xy + y$,

$$P_y = 2x + 1.$$

Conservativeness requires $Q_x = P_y$, so integrating with respect to x gives

$$Q(x, y) = x^2 + x + h(y).$$

Step 2: Apply the boundary data At $x = 0$,

$$Q(0, y) = h(y) = e^y.$$

Thus

$$\boxed{Q(x, y) = x^2 + x + e^y}.$$

Step 3: Find a potential Integrating $f_x = P$ yields

$$f = x^2y + xy + g(y).$$

Then

$$f_y = x^2 + x + g'(y) = Q$$

requires $g'(y) = e^y$, so

$$\boxed{f(x, y) = x^2y + xy + e^y + C}.$$

Verification Its gradient is $\langle 2xy + y, x^2 + x + e^y \rangle$, and setting $x = 0$ in the second component gives exactly e^y .