

**Question 8**

Let  $\mathbf{F}$  be continuous on a region where the indicated curves lie. Prove that the following statements are equivalent:

1. line integrals of  $\mathbf{F}$  are path independent;
2.  $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed curve  $C$ .

**Tasks**

1. Prove path independence implies the closed-loop condition.
2. Prove the converse using two paths with common endpoints.
3. Track orientation carefully in the concatenated curve.

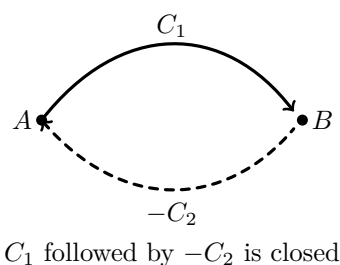
## Question 8 – Solution

**Strategy.** A closed curve can be compared with a constant path, while two endpoint-sharing paths form a closed curve when one is reversed.

**Step 1: Path independence  $\Rightarrow$  zero loops** Let a closed curve start and end at  $A$ . Compare it with the constant path at  $A$ , whose line integral is zero. Path independence gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

**Step 2: Zero loops  $\Rightarrow$  path independence** Let  $C_1$  and  $C_2$  both run from  $A$  to  $B$ . Follow  $C_1$  from  $A$  to  $B$ , then follow  $-C_2$  back to  $A$ . This concatenation is closed.



By the assumed closed-loop condition,

$$0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore

$$\boxed{\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}},$$

which is path independence.

**Verification** The reversal is essential: following  $C_2$  in its original direction would begin at  $A$ , not at the terminal point  $B$  of  $C_1$ , and would not close the loop.