

Question 7

Suppose f and g are potential functions for the same continuous vector field $\mathbf{F}$ on a connected open region D .

Tasks

1. Prove that $f - g$ is constant on D .
2. Explain why connectedness is needed.
3. State the resulting uniqueness principle for potential functions.

Question 7 – Solution

Strategy. Subtract the two gradient equations and use paths inside the connected open region.

Step 1: Subtract gradients Since $\nabla f = \mathbf{F} = \nabla g$,

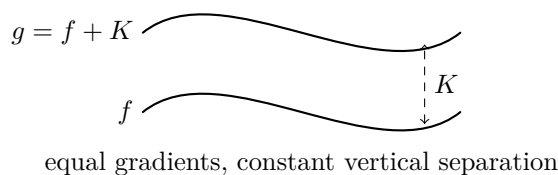
$$\nabla(f - g) = \mathbf{0}.$$

Let $A, B \in D$. An open connected region in $\mathbb{R}^n$ is path connected, so choose a piecewise smooth curve C in D from A to B . Then

$$(f - g)(B) - (f - g)(A) = \int_C \nabla(f - g) \cdot d\mathbf{r} = 0.$$

Thus $(f - g)(B) = (f - g)(A)$ for every pair, and

$$\boxed{f - g = K \text{ on } D}.$$



Step 2: Why connectedness matters On separate components, the same zero gradient permits a different constant on each component because no path in D joins them.

Step 3: Uniqueness principle A potential on a connected region is unique up to one additive constant. Adding a constant changes function values but leaves every partial derivative unchanged.

Verification If $g = f + K$, then $\nabla g = \nabla f + \nabla K = \nabla f$, confirming that every additive constant really does produce another potential.