

Question 6

Determine whether the three-dimensional field

$$\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2y, x + 2z \rangle$$

is conservative on $\mathbb{R}^3$. If so, find a potential and use it to evaluate the integral from $(0, 0, 0)$ to $(1, 2, 3)$.

Tasks

1. Check the three pairwise cross-partial relations.
2. Construct and verify a potential.
3. Evaluate the requested line integral by endpoints.

Question 6 – Solution

Strategy. Verify compatibility of all three components, then integrate successively while retaining correction functions.

Step 1: Compatibility Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$P_y = 2x = Q_x, \quad P_z = 1 = R_x, \quad Q_z = 0 = R_y.$$

On the simply connected domain $\mathbb{R}^3$, these identities show the field is conservative.

Step 2: Construct f Integrating $f_x = P$ gives

$$f = x^2y + xz + g(y, z).$$

From $f_y = x^2 + g_y = Q$, we get $g_y = 2y$, so $g = y^2 + h(z)$. Finally,

$$f_z = x + h'(z) = R = x + 2z,$$

so $h = z^2 + C$. Thus

$$\boxed{f = x^2y + xz + y^2 + z^2 + C}.$$

Step 3: Endpoint evaluation

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 3) - f(0, 0, 0) = 2 + 3 + 4 + 9 = \boxed{18}.$$

Verification Differentiating the boxed potential returns all three original components; the additive constant cancels from the endpoint difference.