

Question 4

On the punctured plane $D = \mathbb{R}^2 \setminus \{(0, 0)\}$, consider

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Investigate whether $\mathbf{F}$ is conservative on D .

Tasks

1. Verify that $P_y = Q_x$ on D .
2. Evaluate the line integral around the counterclockwise unit circle.
3. Reconcile the two results using the geometry of the domain.

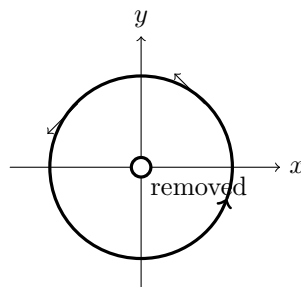
Question 4 – Solution

Strategy. The cross-partial test needs a suitable domain; use a closed-loop integral to detect the hole.

Step 1: Cross partials For $(x, y) \neq (0, 0)$,

$$P_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad Q_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus the local equality holds throughout D .



$\mathbb{R}^2 \setminus \{(0, 0)\}$ has a hole

Step 2: Test a closed circle Set $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = \boxed{2\pi}.$$

A conservative field must have zero integral around every closed curve. Therefore

$$\boxed{\mathbf{F} \text{ is not conservative on } D}.$$

Step 3: Reconcile The punctured plane is not simply connected: the unit circle encloses the missing origin and cannot contract to a point within D . Consequently $P_y = Q_x$ is not sufficient here.