

**Question 3**

For a real constant  $a$ , let

$$\mathbf{F}_a(x, y) = \langle ay + 2xy, 3x + x^2 \rangle.$$

Determine the value of  $a$  for which  $\mathbf{F}_a$  is conservative on  $\mathbb{R}^2$ , and find a potential for that value.

**Tasks**

1. Use the mixed-partial condition to determine  $a$ .
2. Construct a potential function.
3. Verify both components and uniqueness of the parameter.

**Question 3 – Solution**

**Strategy.** Make the cross-partial identity hold for every point, not just at a selected point.

**Step 1: Determine the parameter** Write  $P = ay + 2xy$  and  $Q = 3x + x^2$ . Then

$$P_y = a + 2x, \quad Q_x = 3 + 2x.$$

Equality for all  $x$  requires

$$\boxed{a = 3}.$$

No other constant can make the two affine expressions identical.

**Step 2: Construct the potential** For  $a = 3$ ,

$$f_x = 3y + 2xy.$$

Integrating with respect to  $x$  gives

$$f = 3xy + x^2y + g(y).$$

Now  $f_y = 3x + x^2 + g'(y)$  must equal  $Q = 3x + x^2$ , so  $g'(y) = 0$ .

$$\boxed{f(x, y) = 3xy + x^2y + C}.$$

**Verification** The gradient is  $\langle 3y + 2xy, 3x + x^2 \rangle$ , exactly  $\mathbf{F}_3$ . The coefficient comparison also proves the value  $a = 3$  is unique.