

Question 2

Consider

$$\mathbf{F}(x, y) = \langle y^2, 2xy + x \rangle$$

on $\mathbb{R}^2$.

Tasks

1. Test whether $\mathbf{F}$ is conservative.
2. Identify the exact obstruction.
3. Explain why no potential function can exist even locally near any point.

Question 2 – Solution

Strategy. A continuously differentiable gradient field must satisfy equality of the cross partials.

Step 1: Compute the cross partials Let $P = y^2$ and $Q = 2xy + x$. Then

$$P_y = 2y, \quad Q_x = 2y + 1.$$

Since $P_y \neq Q_x$ at every point, the necessary condition fails.

$\mathbf{F}$ is not conservative on $\mathbb{R}^2$.

Step 2: Identify the contradiction If $\mathbf{F} = \nabla f$ for a twice continuously differentiable f , then

$$f_x = P, \quad f_y = Q,$$

and hence $f_{xy} = P_y$ and $f_{yx} = Q_x$. Equality of mixed partials would require $2y = 2y + 1$, which is impossible.

Step 3: Local conclusion The mismatch is the constant 1 everywhere, not merely at an isolated point. Every open neighborhood contains points where the necessary equality fails, so no local potential exists on any such neighborhood.

Verification Integrating P gives $f = xy^2 + g(y)$, whose y -derivative is $2xy + g'(y)$. Matching Q would require $g'(y) = x$, impossible because g' cannot depend on x .