

Question 10

Let f have continuous first partial derivatives on an open region containing a smooth curve $\mathbf{r}(t)$, $a \leq t \leq b$.
Prove

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Tasks

1. Rewrite the line integral using the parametrization.
2. Apply the multivariable chain rule.
3. Finish with the one-variable Fundamental Theorem of Calculus and identify the hypotheses used.

Question 10 – Solution

Strategy. Show that the vector-field integrand is exactly the ordinary derivative of the composite function $f \circ \mathbf{r}$.

Step 1: Parameter form Write $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. By definition,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Expanding the dot product gives

$$\int_a^b (f_x x'(t) + f_y y'(t) + f_z z'(t)) dt,$$

where the partial derivatives are evaluated at $\mathbf{r}(t)$.

Step 2: Apply the chain rule The multivariable chain rule states

$$\frac{d}{dt} f(\mathbf{r}(t)) = f_x x'(t) + f_y y'(t) + f_z z'(t).$$

Thus the line integral is $\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt$.

Step 3: Apply the one-variable theorem

$$\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = \boxed{f(\mathbf{r}(b)) - f(\mathbf{r}(a))}.$$

The smoothness of $\mathbf{r}$ and continuity of the first partial derivatives make the chain-rule derivative continuous and integrable. The same proof applies piece by piece to a piecewise smooth curve, with intermediate endpoint values canceling.

Verification The right side reverses sign when a and b are interchanged, matching orientation reversal of the line integral.