

Question 8

Let

$$f(x, y, z) = xyz + \frac{1}{2}(x^2 + y^2 + z^2).$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ for any smooth space curve from $A = (1, -1, 0)$ to $B = (2, 1, 3)$.

Tasks

1. Compute $f(A)$ and $f(B)$ carefully.
2. Apply the Fundamental Theorem for Line Integrals.
3. Verify that the associated vector field is three-dimensional by computing ∇f .

Question 8 – Solution

Strategy. The spatial geometry of C is unnecessary because the integrand is the gradient of the supplied scalar function.

Step 1: Initial value

$$f(1, -1, 0) = (1)(-1)(0) + \frac{1}{2}(1 + 1 + 0) = 1.$$

Step 2: Terminal value

$$f(2, 1, 3) = (2)(1)(3) + \frac{1}{2}(4 + 1 + 9) = 6 + 7 = 13.$$

Thus

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 13 - 1 = 12}.$$

Step 3: Display the field Differentiating componentwise gives

$$\nabla f = \langle yz + x, xz + y, xy + z \rangle.$$

This confirms that the line integral uses all three coordinate components, even though no parametrization is needed.

Verification The quadratic contribution changes from 1 to 7, a gain of 6, and the product contribution changes from 0 to 6, another gain of 6. The total change is 12.