

Question 7

Let

$$f(x, y) = x^3 - 3xy + y^2.$$

A piecewise smooth path C travels from $A = (0, 0)$ to $(1, 0)$, then to $(1, 1)$, and finally to $B = (2, 1)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ using the theorem.
2. Show how the endpoint changes telescope across the three pieces.
3. Explain why the corners do not invalidate the theorem.

Question 7 – Solution

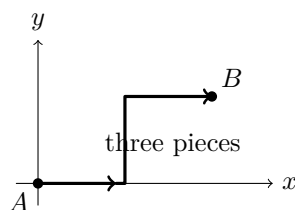
Strategy. Apply the theorem to the whole piecewise smooth path, then expose the same cancellation piece by piece.

Step 1: Whole-path calculation

$$f(A) = 0, \quad f(B) = 2^3 - 3(2)(1) + 1^2 = 8 - 6 + 1 = 3.$$

Therefore

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = 3}.$$



Step 2: Telescope Let $P = (1, 0)$ and $Q = (1, 1)$. Then

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= [f(P) - f(A)] + [f(Q) - f(P)] + [f(B) - f(Q)] \\ &= f(B) - f(A) = 3. \end{aligned}$$

Every intermediate potential value appears once positively and once negatively.

Step 3: Corners Smoothness is required on each segment, not at the finitely many joining corners. Adding the valid integral on each piece yields the integral on the piecewise smooth path.

Verification Direct values $f(P) = 1$ and $f(Q) = -1$ give contributions 1, -2 , 4, whose sum is 3.