

Question 5

Let $f(x, y) = x^2y$ and let C be the quarter unit circle

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ in two ways.

Tasks

1. Use the Fundamental Theorem for Line Integrals.
2. Evaluate directly from $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.
3. Identify the chain-rule antiderivative in the direct calculation.

Question 5 – Solution

Strategy. First use the endpoints, then verify that the parameter integrand is the derivative of $f(\mathbf{r}(t))$.

Step 1: Endpoint method The curve begins at $(1, 0)$ and ends at $(0, 1)$. Both endpoint values are zero:

$$f(1, 0) = 0, \quad f(0, 1) = 0.$$

Therefore the theorem gives $\boxed{0}$.

Step 2: Direct method Since $\nabla f = \langle 2xy, x^2 \rangle$,

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle 2 \cos t \sin t, \cos^2 t \rangle \cdot \langle -\sin t, \cos t \rangle \\ &= -2 \cos t \sin^2 t + \cos^3 t.\end{aligned}$$

This is

$$\frac{d}{dt}(\cos^2 t \sin t) = \frac{d}{dt}f(\mathbf{r}(t)).$$

Hence

$$\int_0^{\pi/2} \frac{d}{dt}(\cos^2 t \sin t) dt = [\cos^2 t \sin t]_0^{\pi/2} = \boxed{0}.$$

Verification The two methods agree, and the displayed antiderivative explicitly exhibits the chain rule behind the theorem.