

Question 9

Let C be the line segment from $(0,0)$ to $(1,2)$ in the field

$$\mathbf{F}(x,y) = \langle x-y, x+y \rangle.$$

Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using both

$$\mathbf{r}_1(t) = \langle t, 2t \rangle, \ 0 \leq t \leq 1, \quad \text{and} \quad \mathbf{r}_2(u) = \langle u^2, 2u^2 \rangle, \ 0 \leq u \leq 1.$$

Tasks

1. Evaluate the integral with each parametrization.
2. Explain why the zero velocity of $\mathbf{r}_2$ at $u = 0$ causes no problem.
3. State the invariance illustrated by the equal answers.

Question 9 – Solution

Strategy. Carry the parameter derivative through both computations; the slower start must be included in dr .

Step 1: First parametrization Here

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle -t, 3t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 2 \rangle.$$

Thus

$$\int_0^1 (-t + 6t) dt = \int_0^1 5t dt = \boxed{\frac{5}{2}}.$$

Step 2: Nonuniform parametrization Now

$$\mathbf{F}(\mathbf{r}_2(u)) = \langle -u^2, 3u^2 \rangle, \quad \mathbf{r}'_2(u) = \langle 2u, 4u \rangle.$$

Therefore

$$\int_0^1 (-2u^3 + 12u^3) du = \int_0^1 10u^3 du = \boxed{\frac{5}{2}}.$$

Step 3: Interpret Although $\mathbf{r}'_2(0) = \mathbf{0}$, the curve pauses only at one endpoint and then traces the segment once with the same orientation. The derivative factor compensates for its varying speed.

Verification Both maps begin at $(0, 0)$, end at $(1, 2)$, and move monotonically along the same segment, so parameter independence requires the matching values obtained above.