

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle, \quad \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Evaluate the field along the space curve.
2. Compute and integrate the dot product.
3. Verify the calculation by expanding $z \, dx + x \, dy + y \, dz$.

Question 5 – Solution

Strategy. Substitute all three coordinates before taking the dot product with the three-dimensional velocity.

Step 1: Field and velocity

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^3, t, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle.$$

Step 2: Dot and integrate

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= t^3 + 2t^2 + 3t^4, \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \left[\frac{t^4}{4} + \frac{2t^3}{3} + \frac{3t^5}{5} \right]_0^1 \\ &= \frac{1}{4} + \frac{2}{3} + \frac{3}{5} = \boxed{\frac{91}{60}}. \end{aligned}$$

Step 3: Differential-form check Along the curve,

$$z \, dx = t^3 \, dt, \quad x \, dy = t(2t \, dt) = 2t^2 \, dt, \quad y \, dz = t^2(3t^2 \, dt) = 3t^4 \, dt.$$

These are exactly the three terms above.

Verification Every term is nonnegative on $[0, 1]$, and the common-denominator sum $(15 + 40 + 36)/60 = 91/60$ confirms the arithmetic.