

Question 10

A wire occupies the upper semicircle

$$x^2 + y^2 = R^2, \quad y \geq 0,$$

and has density $\delta(x, y) = 1 + y/R$, where $R > 0$.

Tasks

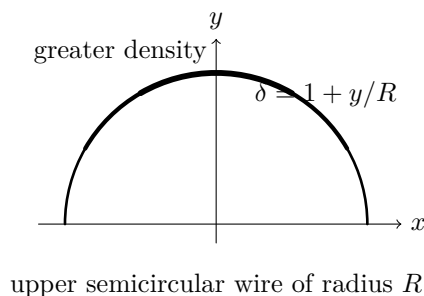
1. Find the wire's mass.
2. Compute its moments about the x - and y -axes.
3. Determine its center of mass and check the result by symmetry and bounds.

Question 10 – Solution

Strategy. Parametrize by polar angle; symmetry immediately predicts that the center of mass lies on the y -axis.

Step 1: Mass Let $x = R \cos \theta$, $y = R \sin \theta$, $0 \leq \theta \leq \pi$, so $ds = R d\theta$ and $\delta = 1 + \sin \theta$. Then

$$M = R \int_0^\pi (1 + \sin \theta) d\theta = \boxed{R(\pi + 2)}.$$



Step 2: Moments The moment about the x -axis is

$$\begin{aligned} M_x &= \int_C y \delta ds = R^2 \int_0^\pi \sin \theta (1 + \sin \theta) d\theta \\ &= R^2 \left(2 + \frac{\pi}{2} \right). \end{aligned}$$

For the y -axis,

$$M_y = R^2 \int_0^\pi \cos \theta (1 + \sin \theta) d\theta = 0.$$

Step 3: Center of mass

$$\bar{x} = \frac{M_y}{M} = 0, \quad \bar{y} = \frac{M_x}{M} = \boxed{\frac{R(\pi + 4)}{2(\pi + 2)}}.$$

Thus

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{R(\pi + 4)}{2(\pi + 2)} \right)}.$$

Verification Left-right symmetry forces $\bar{x} = 0$. Also $0 < (\pi + 4)/[2(\pi + 2)] < 1$, so $0 < \bar{y} < R$, placing the center of mass inside the semicircle's vertical range.