

Question 8

Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$. Find the average value of

$$f(x, y) = x^2$$

along C with respect to arc length.

Tasks

1. Compute $\int_C x^2 ds$ and the length of C .
2. Find the arc-length average value.
3. Identify all points on C where x^2 equals that average.

Question 8 – Solution

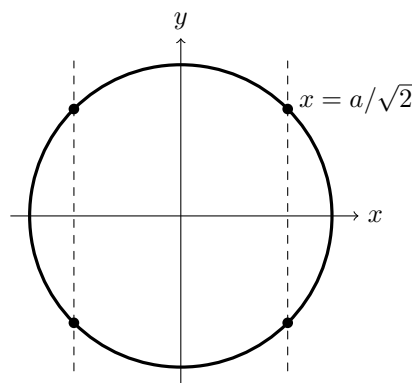
Strategy. Parametrize the circle and divide the scalar line integral by total arc length.

Step 1: Integral and length Use

$$\mathbf{r}(\theta) = \langle a \cos \theta, a \sin \theta \rangle, \quad 0 \leq \theta \leq 2\pi.$$

Then $ds = a d\theta$, so

$$\int_C x^2 ds = a^3 \int_0^{2\pi} \cos^2 \theta d\theta = \pi a^3, \quad L = 2\pi a.$$



$$x^2 + y^2 = a^2 \text{ (scaled sketch)}$$

Step 2: Average value

$$f_{\text{avg}} = \frac{1}{L} \int_C f ds = \frac{\pi a^3}{2\pi a} = \boxed{\frac{a^2}{2}}.$$

Step 3: Points attaining the average We need $x^2 = a^2/2$, so $x = \pm a/\sqrt{2}$. The circle equation then gives $y = \pm a/\sqrt{2}$, independently. Thus the four points are

$$\boxed{\left(\frac{\pm a}{\sqrt{2}}, \frac{\pm a}{\sqrt{2}} \right)},$$

with all four sign combinations.

Verification Symmetry gives equal averages for x^2 and y^2 . Their sum is constantly a^2 on C , so each average must be $a^2/2$.