

Question 7

A wire follows the first-quadrant astroid arc

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad 0 \leq t \leq \frac{\pi}{2},$$

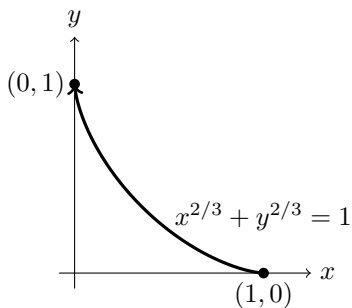
and has linear density $\delta(x, y) = x + y$. Find its mass.

Tasks

1. Derive and simplify the speed on the given interval.
2. Express the density in terms of t and evaluate the mass.
3. Check positivity and endpoint behavior.

Question 7 – Solution

Strategy. Differentiate the astroid parametrization; in the first quadrant the absolute values in the speed simplify.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t \rangle.$$

Because $\sin t, \cos t \geq 0$,

$$\begin{aligned} |\mathbf{r}'(t)| &= 3 \sin t \cos t \sqrt{\cos^2 t + \sin^2 t} \\ &= 3 \sin t \cos t. \end{aligned}$$

Step 2: Mass Along the wire, $\delta = \cos^3 t + \sin^3 t$. Hence

$$\begin{aligned} M &= \int_0^{\pi/2} 3(\cos^3 t + \sin^3 t) \sin t \cos t \, dt \\ &= 3 \left(\int_0^{\pi/2} \cos^4 t \sin t \, dt + \int_0^{\pi/2} \sin^4 t \cos t \, dt \right) \\ &= 3 \left(\frac{1}{5} + \frac{1}{5} \right) = \boxed{\frac{6}{5}}. \end{aligned}$$

Verification Density and speed are nonnegative. The speed vanishes only at the two cusps, which does not affect the integral; the finite positive mass $6/5$ is therefore consistent.