

Question 10

Let f be continuously differentiable on a region D of area 12, and suppose

$$1 \leq |\nabla f(x, y)| \leq 2 \quad \text{throughout } D.$$

Let S be the area of the graph $z = f(x, y)$ above D .

Tasks

1. Derive the sharpest bounds guaranteed for S .
2. Prove the bounds from the surface-area integral.
3. Show that both endpoints can occur.

Question 10 – Solution

Strategy. Apply the monotonicity of $\sqrt{1+t^2}$ to the pointwise gradient bounds, then integrate.

Step 1: Pointwise comparison From $1 \leq |\nabla f| \leq 2$,

$$\sqrt{2} = \sqrt{1+1^2} \leq \sqrt{1+|\nabla f|^2} \leq \sqrt{1+2^2} = \sqrt{5}.$$

Step 2: Integrate The graph area is

$$S = \iint_D \sqrt{1+|\nabla f|^2} dA.$$

Integrating the pointwise inequalities and using $\text{area}(D) = 12$ yields

$$\boxed{12\sqrt{2} \leq S \leq 12\sqrt{5}}.$$

Step 3: Sharpness The lower endpoint occurs for any plane with constant gradient magnitude 1, for example $f(x, y) = x$. The upper endpoint occurs for a plane with constant gradient magnitude 2, for example $f(x, y) = 2x$. Indeed,

$$S_{f=x} = 12\sqrt{1+1} = 12\sqrt{2}, \quad S_{f=2x} = 12\sqrt{1+4} = 12\sqrt{5}.$$

Therefore no narrower interval can be guaranteed from the supplied data.