

Question 7

Find the area of the surface

$$z = xy$$

above the quarter-disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

Tasks

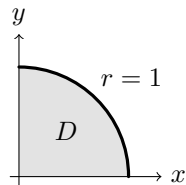
1. Show that the surface-area factor is radial even though $z = xy$ is not.
2. Evaluate the polar integral.
3. Verify the angular and radial bounds.

Question 7 – Solution

Strategy. Differentiate first: the two gradient components exchange x and y , so their squares combine to r^2 .

Step 1: Gradient and projection

$$f_x = y, \quad f_y = x, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}.$$



The quarter-disk has $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$.

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{\pi/2} \int_0^1 \sqrt{1 + r^2} r \, dr \, d\theta \\ &= \frac{\pi}{2} \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^1 = \boxed{\frac{\pi}{6}(2\sqrt{2} - 1)}. \end{aligned}$$

Verification The angular width is one quarter of a full turn. The factor ranges from 1 to $\sqrt{2}$, so the result must lie between the projection area $\pi/4$ and $\sqrt{2}\pi/4$; it does.