

Question 4

Use a surface-area integral to find the area of the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}.$$

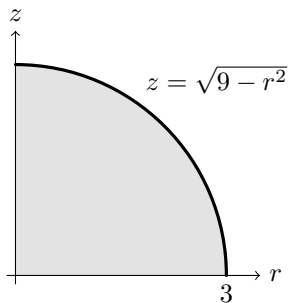
Tasks

1. Derive the surface-area factor over the open disk $x^2 + y^2 < 9$.
2. Evaluate the resulting improper integral.
3. Explain why the boundary singularity is integrable.

Question 4 – Solution

Strategy. Integrate over disks of radius $b < 3$, then take the limit as $b \rightarrow 3^-$.

Step 1: Area factor



With $r^2 = x^2 + y^2$,

$$f_x = -\frac{x}{\sqrt{9-r^2}}, \quad f_y = -\frac{y}{\sqrt{9-r^2}},$$

so

$$\sqrt{1 + f_x^2 + f_y^2} = \frac{3}{\sqrt{9-r^2}}.$$

Step 2: Improper integral

$$\begin{aligned} S &= \lim_{b \rightarrow 3} \int_0^{2\pi} \int_0^b \frac{3r}{\sqrt{9-r^2}} dr d\theta \\ &= \lim_{b \rightarrow 3} 6\pi \left(3 - \sqrt{9-b^2} \right) = \boxed{18\pi}. \end{aligned}$$

Verification The factor diverges as $r \rightarrow 3^-$ because the sphere becomes vertical, but its antiderivative has a finite limit. The result also equals half the full sphere area $4\pi(3^2) = 36\pi$.